COUPLED CONTINUOUS AND DISCONTINUOUS GALERKIN FINITE ELEMENT METHODS FOR COASTAL AND OCEAN MODELING

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Abstract. Finite element methodologies have long been used in modeling circulation in coastal and oceanic waters. Many codes in wide use today, e.g., the QUODDY and ADCIRC simulators, are based on continuous Galerkin (CG) finite elements for both elevation and momentum. Because continuous elements applied to the primitive continuity equation are subject to spurious oscillations and/or phase errors, these codes utilize a reformulation of the continuity equation into a so-called generalized wave continuity equation (GWCE). Moreover, these finite element codes are not "locally conservative," that is, they do not produce mass conservative fluxes. In recent years, finite element methods based on discontinuous basis functions have been developed (the discontinuous Galerkin or DG method). These methods are locally conservative and generally provide sharper resolution of fronts with minimal oscillation and phase error. However, these methods can involve more degrees of freedom than their continuous counterparts. In this paper, we will present a multi-algorithmic finite element strategy based on utilizing both continuous and discontinuous Galerkin methods. We will focus on recent developments in the ADCIRC simulator, where we have replaced the GWCE CG formulation with a DG formulation based on the primitive continuity equation. Numerical results will be presented for this new approach.

1. Introduction. Many standard coastal and ocean models are based on the Galerkin finite element method. For example, the ADCIRC [13] and QUODDY [10] simulators use finite elements with continuous, piecewise linear approximations of elevation and velocity. Because of instabilities that can arise when using these types of approximations for pure hyperbolic equations, these codes utilize a reformulation of the primitive continuity equation into a Generalized Wave Continuity Equation (GWCE) [12]. While the GWCE provides accurate solutions for many types of smooth flow problems, the mass conservation inherent in the primitive continuity equation is lost in this formulation. Moreover, it can be somewhat cumbersome to implement.

In recent years, Galerkin finite element methods based on discontinuous approximating spaces have been proposed for many different types of partial differential equations [5]. These methods can be traced back to the 1970s, where they were developed for elliptic and parabolic equations [11, 2, 15]. They were developed extensively in the 1980's and 1990's for hyperbolic equations [8, 7, 6, 4, 9]. Discontinuous Galerkin (DG) methods possess several interesting features which may be useful in certain applications. First, they easily allow for varying the polynomial order of approximation from one element to the next. They allow for very general meshes, including non-conforming meshes, without hanging nodes or the need for a mortar space. One can also build stability post-processing into the methods for minimizing oscillations in the presence of high gradients. Finally, the methods are "locally conservative," that is, they are based on satisfying conservation principles element-by-element. One potential disadvantage of DG methods over continuous Galerkin methods is that the degrees of freedom in a DG method are associated with elements, while in the continuous Galerkin method they are associated with nodes. Hence, for the same degree approximating spaces on the same grid, there may be many more degrees of freedom needed for the DG method.

Recently, the authors and collaborators at The University of Texas at Austin have investigated the use of DG methods for the numerical solution of the shallow water equations, resulting in a simulator which we refer to as UTBEST [1, 3]. This code uses a full DG formulation for both elevation and velocity. It has proven to work well for both smooth flows and supercritical flows, for example,

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through constricted channels. In particular, the code produces solutions which locally conserve both mass and momentum.

As pointed out above, for the same mesh, the DG formulation in UTBEST requires substantially more degrees of freedom than its CG counterpart. In particular, the velocity solution is much more expensive to compute, especially as we extend to three dimensional flows. For this and other reasons, we are investigating the coupling of the DG and CG formulations used in UTBEST and in ADCIRC. Our first attempt at this coupling involves using the DG method for the primitive continuity equation, while still using a CG method for momentum. In this way, we avoid the use of the GWCE, and obtain solutions which are locally and globally mass conservative. It is this coupling that will be the focus of this paper.

The paper is organized as follows. In the next section, we state the problem to be considered and formulate the coupled DG-CG method. Then, in section three, some numerical results are given.

2. Problem definition and numerical approach. We will consider the depth-averaged shallow water equations:

$$\frac{\partial \xi}{\partial t} + \nabla \cdot (\mathbf{u}H) = 0, \tag{1}$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \tau_{bf}\mathbf{u} + f_c \mathbf{k} \times \mathbf{u} + g\nabla\xi - \frac{\nu_T}{H}\Delta(H\mathbf{u}) = \frac{1}{H}\mathbf{F}. \tag{2}$$

(1) represents the conservation of mass and is also referred to as the primitive continuity equation; (2) represents the conservation of momentum in non-conservative form. In the above equations, ξ represents the deflection of the air-liquid interface from the mean sea level, $H = h_b + \xi$ represents the total fluid depth, and h_b is the bathymetric depth, $\mathbf{u} = $ (u,v) is the depth averaged horizontal velocity field, f_c is the Coriolis parameter resulting from the earth's rotation, $\mathbf{k}$ is the local vertical vector, g is the gravitational acceleration, τ_{bf} is the bottom friction coefficient and ν_T is the depth averaged turbulent viscosity. We have used the form of the diffusion term as given in [13]. In addition to the above described phenomena, often we need to include the effects of surface wind stress, variable atmospheric pressure and tidal potentials which are expressed through the body force $\mathbf{F}$ [14].

The above equations are defined over a spatial domain Ω and for $t > 0$. Initial conditions for ξ and $\mathbf{u}$ must be specified. Boundary conditions usually consist of specified flow (river, land) and/or specified elevation (open sea).

Let $\{\mathcal{T}_h\}_{h>0}$ denote a regular family of triangular finite element partitions of Ω such that no triangle Ω_e crosses $\partial\Omega$. Let h_e denote the element diameter and h the maximal element diameter. We will use the $L^2(R)$ inner product notation $(\cdot,\cdot)_R$ for domains $R \in \mathbb{R}^d$, and the notation $\langle u, v\rangle_R$ to denote integration over $d-1$ dimensional surfaces. On Ω_e, we define the approximating space $\mathcal{P}^k(\Omega_e)$, where $\mathcal{P}^k$ denotes the set of complete polynomials of degree k. Let

$$W_h = \{w : w|_{\Omega_e} \in \mathcal{P}^k(\Omega_e)\}.$$

We approximate $\mathbf{u}$ using a standard continuous Galerkin finite element method applied to (2); in particular, we use continuous piecewise linears defined on triangles. We will not go through this formulation here, as it is well known. Let $\mathbf{u}_h$ denote the approximation obtained in this way. For approximating ξ we use a DG method, which we now describe.

Multiplying (1) by a test function w and integrating over an element Ω_e, we find

$$(\frac{\partial \xi}{\partial t}, w)_{\Omega_e} - (\mathbf{u}H, \nabla w)_{\Omega_e} + \langle \mathbf{u} \cdot \mathbf{n}_e H, w\rangle_{\partial\Omega} = 0, \tag{3}$$

where $\mathbf{n}_e$ is the outward unit normal to $\partial\Omega_e$. Approximations $\xi \approx \xi_h \in W_h$, $H_h = \xi_h + h_b$ are obtained from

$$(\frac{\partial \xi_h}{\partial t}, w_h)_{\Omega_e} - (\mathbf{u}_h H_h, \nabla w_h)_{\Omega_e} + \langle \mathbf{G} \cdot \mathbf{n}_e, w_h\rangle_{\partial\Omega} = 0, \quad w_h \in W_h. \tag{4}$$

The term $\mathbf{G} \cdot \mathbf{n}_e$ is an approximation to $H\mathbf{u} \cdot \mathbf{n}_e$, that is, $\mathbf{G} \approx \mathbf{u}H$. In order to treat this term correctly, we must recognize that (1)-(2) is a coupled system. For a complete description of how this term is handled, we refer the reader to [1, 3], where we describe the Roe flux approximation to the system of conservation laws (1)-(2). $\mathbf{G} \cdot \mathbf{n}_e$ is taken to be the first term in this approximation. We remark that this approximation is continuous across element boundaries.

We have presented our coupled DG/CG method in continuous time. At the moment, we are using a sequential time stepping approach. Given $\mathbf{u}_h$ at time t^n, we march (4) forward explicity in time to obtain ξ_h at time $t^{n+1} = t^n + \Delta t$. We then use this approximation to march (2) forward explicitly to obtain $\mathbf{u}_h$ at t^{n+1}.

Finally, we remark that (4) is locally conservative in the following sense. Letting $w_h = 1$ in (4), we obtain

$$\int_{\Omega_e} \frac{\partial \xi_h}{\partial t} dx + \int_{\partial \Omega_e} \mathbf{G} \cdot \mathbf{n}_e ds = 0. \tag{5}$$

Thus, the change in elevation is balanced by fluxes through the boundary of the element. Moreover, as $\mathbf{G} \cdot \mathbf{n}_e$ is continuous, we can sum over all elements Ω_e to obtain a statement of global conservation of mass over all of Ω.

3. Numerical Results. In this section, we present numerical results for two test cases. The first test case is a quarter annular shaped harbor, and the second test case is a section of the Mississippi River in the coastal region of Louisiana. In both of these test cases we have used piecewise constant ($k = 0$) approximations in the DG method.

The first test case models tidal flow in a harbor. A coarse finite element mesh consisting of 50 elements and 36 nodes was used, with bathymetry varying from 59 to 13 meters, as seen in Figure 1. An open sea boundary condition is enforced at the outer circular boundary. The other three boundaries are treated as land boundaries. For this test case, we compare solutions generated using the ADCIRC, UTBEST and coupled formulation described above, at three different nodes, labeled 1, 2 and 3 in Figure 1. In Figure 2, we have compared elevation solutions versus time at the three nodes. The simulation was carried out to 10 days with a time step of 86.4 seconds, or 10000 time steps. As can be seen in the figures excellent agreement is seen between the three codes. The solutions for the x component of velocity are given in Figure 3. Here, we see that the solutions agree very well except at node 2, which is close to a land boundary. This discrepancy is most likely due to the different ways that ADCIRC and UTBEST enforce the no-flow condition at the land boundary.

The second test case involves a section of the Mississippi River near the coast of Louisiana, see Figure 4. A finite element mesh consisting of 35281 elements and 19616 nodes was used in this simulation. The length of simulation was 2 days. This case involves a substantial amount of wetting and drying. The bathymetry varies from about 50 meters to -9 meters. In Figure 5, we show a zoom into part of the river, with a contour of bathymetry in this region. For this case, we compare elevation and velocity solutions vs. time between ADCIRC and the CG/DG method at three nodes, labeled 1, 2 and 3 in Figure 4. In Figure 6, we show the elevation solutions at the three nodes. As in the quarter annular test case, fairly good agreement is seen between the two solutions, with maximum differences on the order of .04 meters. In Figure 7, we show x-component velocity solutions at the three nodes. Here the differences are more pronounced, but still maximum differences are relatively small, on the order of .06 m/sec.

4. Conclusions. In this paper, we have presented a new approach for shallow water simulation, based on using a discontinuous Galerkin finite element method for elevation, coupled to a continuous Galerkin method for velocity. Numerical results show that this approach is promising, and gives locally mass conservative solutions. Further tests will be carried out to fully ascertain the robustness of this approach.
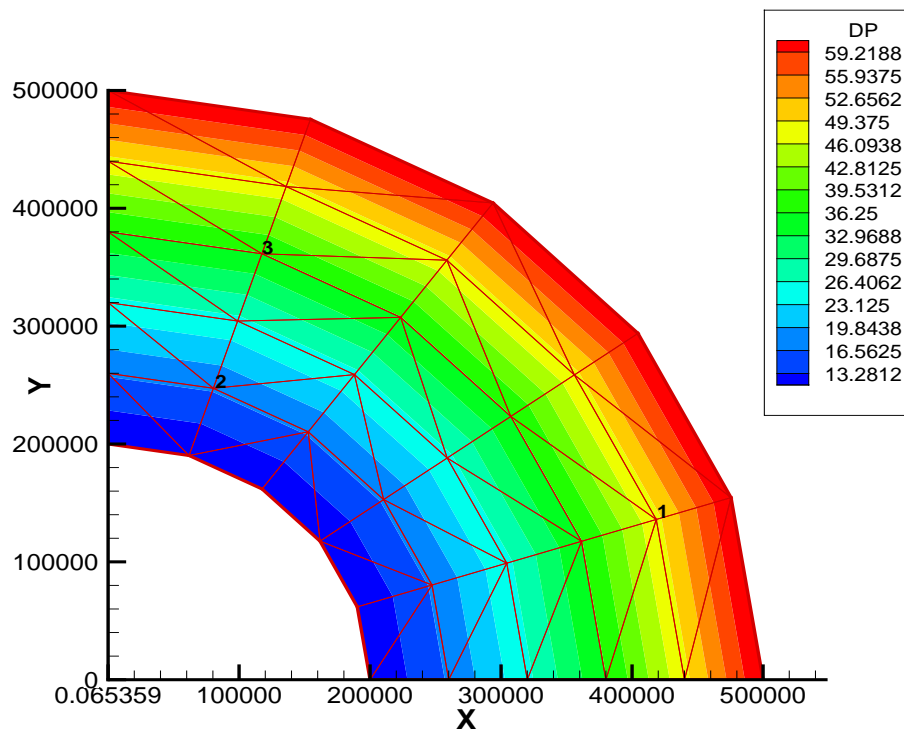
Fɪɢ. 1. *Finite element mesh for quarter annular harbor.*

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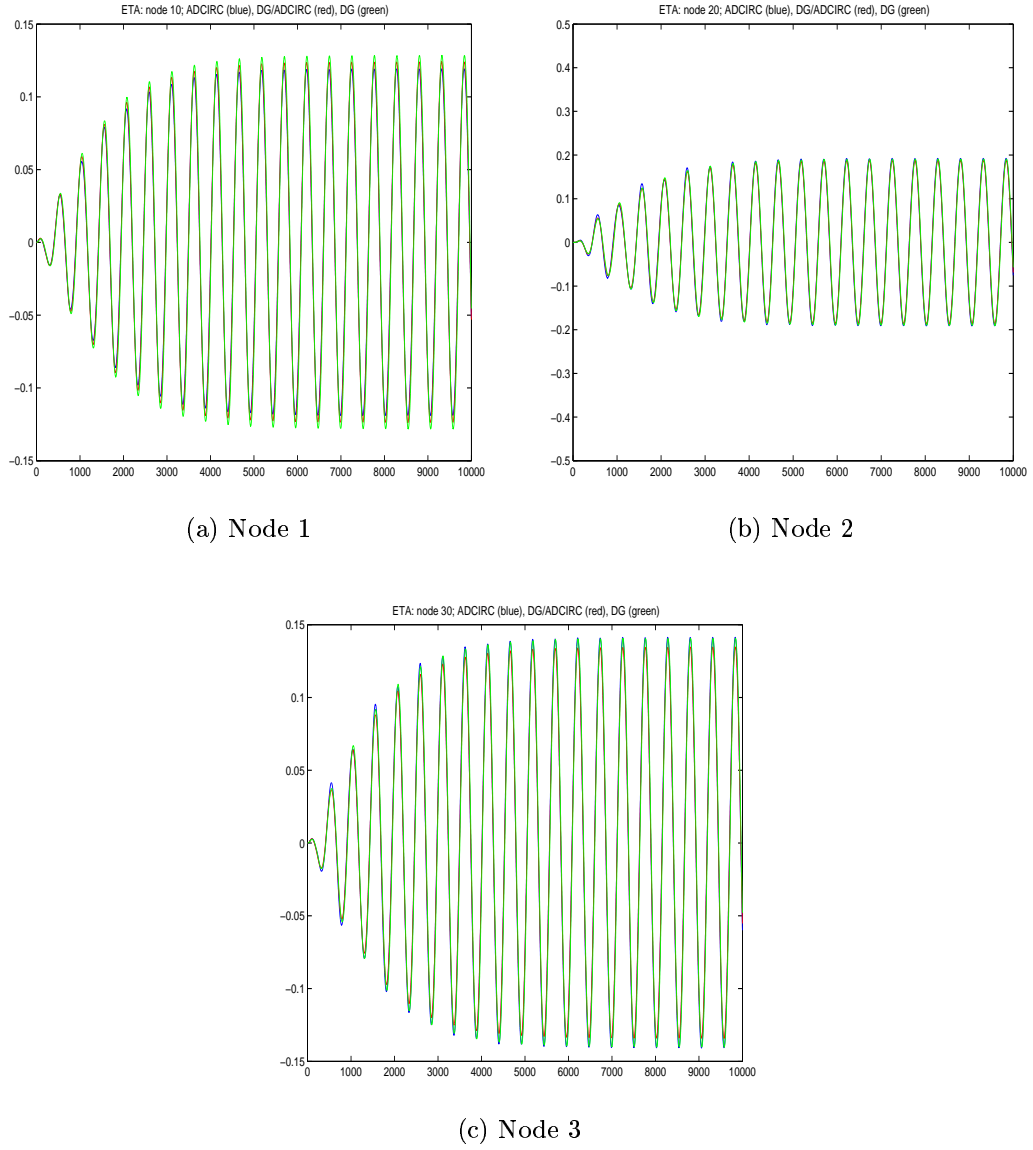[8] ——, *The Runge-Kutta local projection P^1-discontinuous Galerkin method for scalar conservation*

(a) Node 1

(b) Node 2

(c) Node 3

Fig. 2. *Comparison of ADCIRC, UTBEST and CG/DG elevation solutions at the three nodes in Figure 1.*

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(a) Node 1

(b) Node 2

(c) Node 3
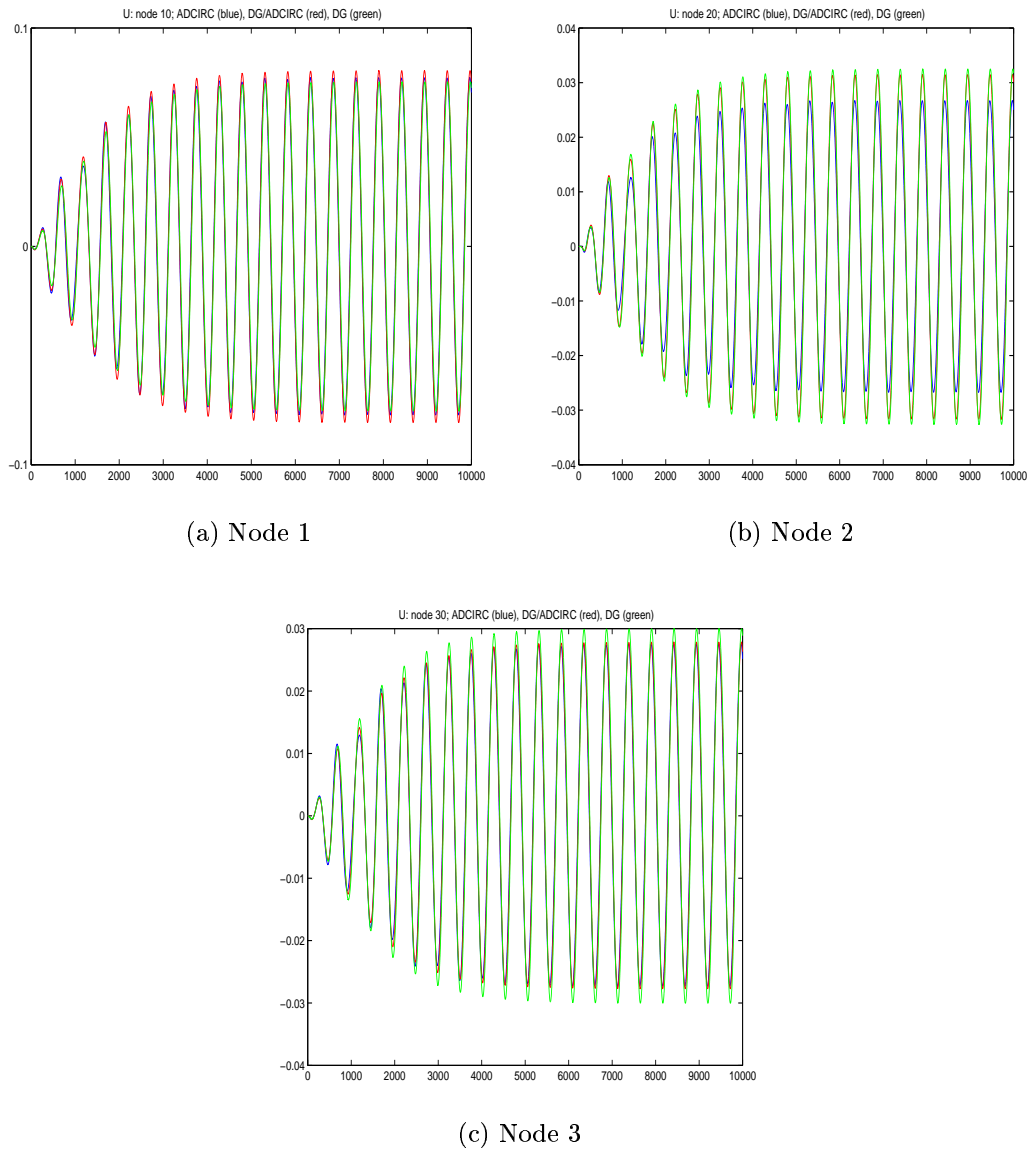
Fig. 3. *Comparison of ADCIRC, UTBEST and CG/DG velocity solutions at the three nodes in Figure 1.*

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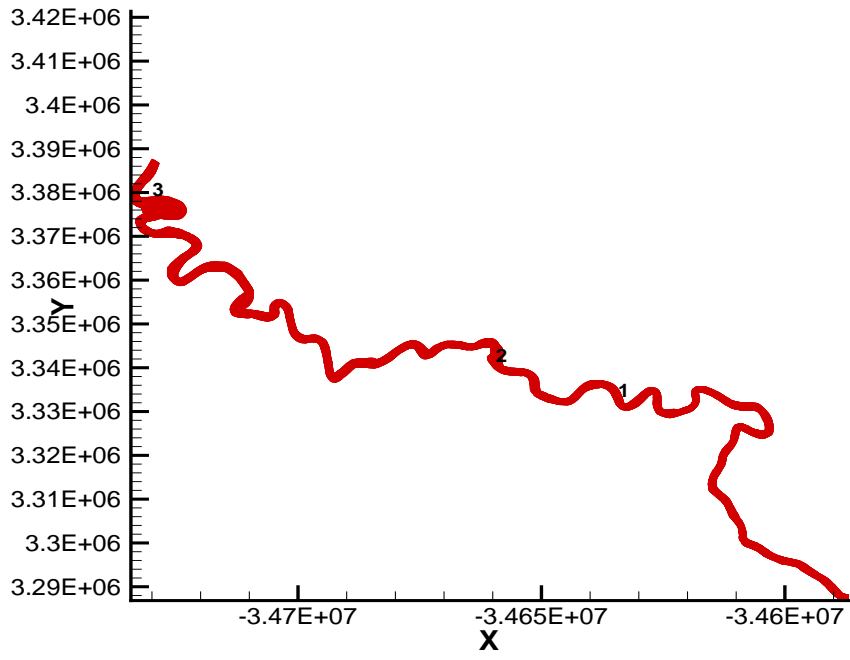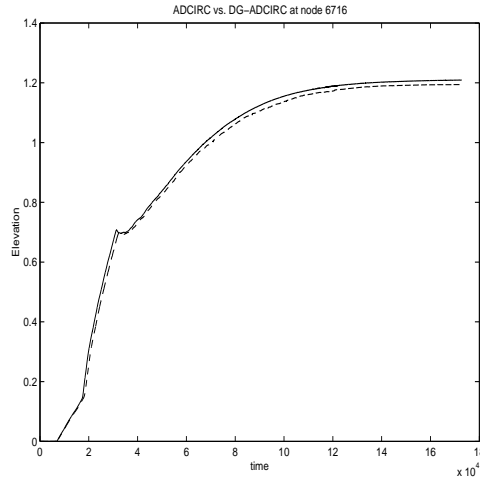
FIG. 4. *Domain for Mississippi River dataset.*
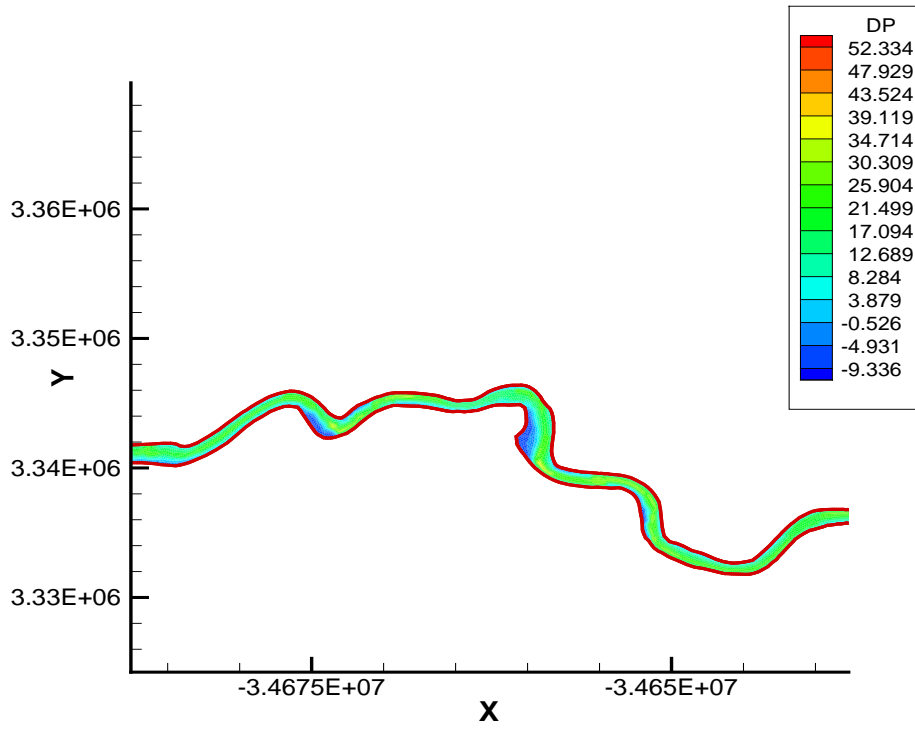
FIG. 5. *Bathymetry for a section of the Mississippi River.*
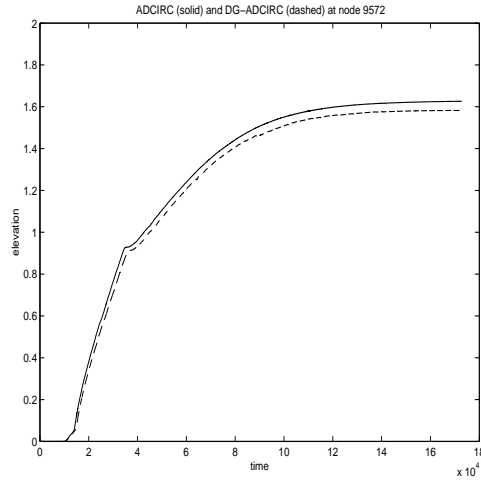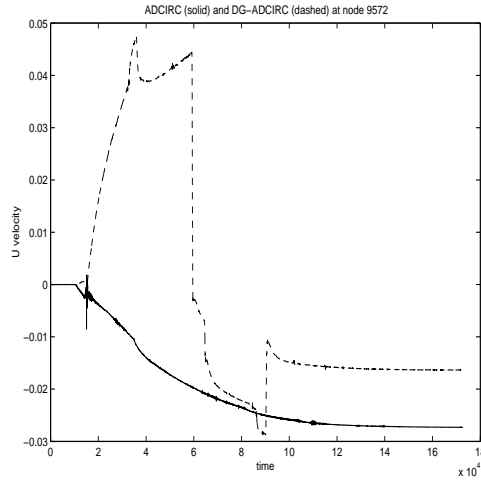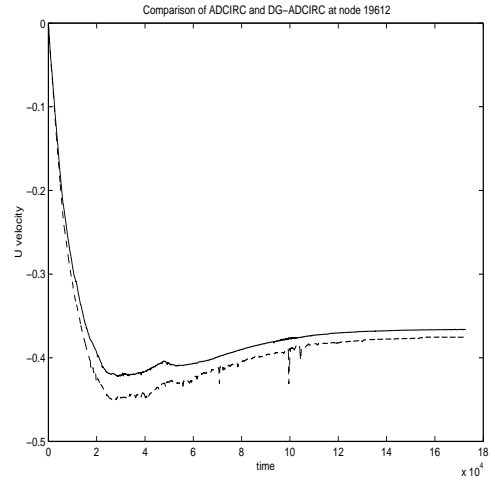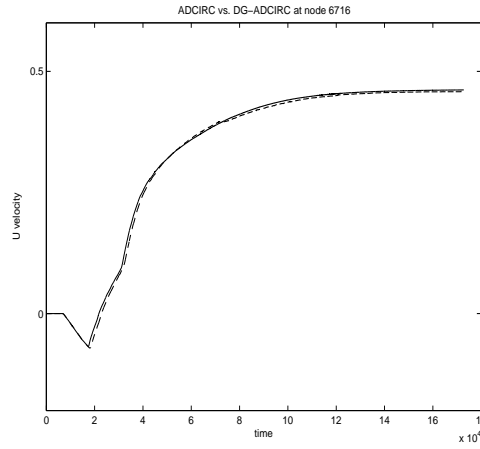
(a) Node 1
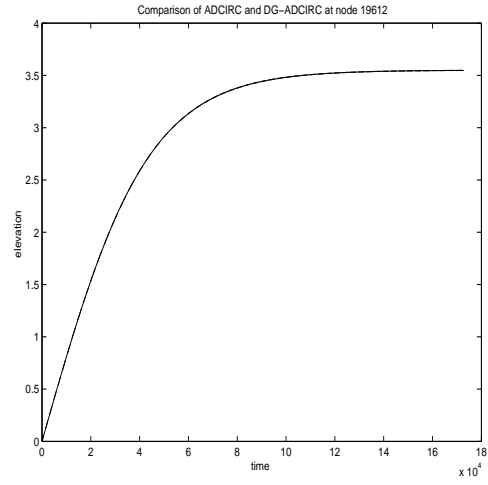
(b) Node 2

(c) Node 3

FIG. 6. *Comparison of ADCIRC (solid) and CG/DG (dashed) elevation solutions at the three nodes in Figure 4.*

(a) Node 1

(b) Node 2

(c) Node 3

FIG. 7. *Comparison of ADCIRC (solid) and CG/DG (dashed) velocity solutions at the three nodes in Figure 4.*